FORMATION FLYING CONTROL OF MULTIPLE SPACECRAFT: GRAPH THEORETIC PROPERTIES AND SWITCHING SCHEMES

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Abstract

We present several results on the leader-following (LF) paradigm in the formation flying of multiple spacecraft in free space. In this direction, ideas from elementary graph theory and linear matrix inequalities are combined with logic-based switching to shed light on the various control designs which are feasible using the leader-following mechanism for various formation scenarios.

Keywords: Formation flying; Linear matrix inequality; Logic-based switching; Graph theory

1 Introduction

1.1

Formation flying (FF) has been identified as an enabling technology for many of the NASA's 21st century missions, among them, the Space Technology 3 and the Terrestrial Planet Finder. Formation flying involves flying a group of spacecraft in a particular pattern while maintaining precise (possibly time varying) relative position, velocity, attitude, and angular velocity, with respect to each other [7], [14]. Since traditional spacecraft control is often concerned with measuring and maintaining the same quantities for a single spacecraft with respect to an inertial reference frame, the analogous FF control and estimation problems are often an order of magnitude more challenging than those encountered traditionally for a single spacecraft [2], [9], [16]. In order to make the FF control problems at least *similar* to the single spacecraft case, an approach based on leader-following has been proposed by Wang and Hadaegh [15]. The basic idea in leader-following (LF) is to designate a particular frame (or multiple frames) in the formation as the reference frame(s) of interest and measure and control the states of the rest of the formation with respect to them.

The purpose of the present paper is twofold. First it is shown that LF can naturally be given a graph theoretic formulation, and by doing so, we are able to obtain non-trivial results on some of its basic properties. We then derive control laws for LF using linear matrix inequalities (LMIs) [4]. These control laws are subsequently employed in the second part of the paper to propose logic-based switching schemes for the formation control.

The outline of the paper is as follows. In §2 the notation and the mathematical preliminaries are presented, among them, certain facts on elementary graph theory, LMIs, hybrid and switching systems. §3 is devoted to leader-following and its graph theoretic interpretation. Simple control laws are then derived in §4. In §5, §6, and §7, the control laws derived in §4 are combined with logic-based switching to propose a hybrid control architecture for leader reassignment, LF capturing (defined subsequently), and dealing with control saturations.

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2 Notation and Preliminaries

2.1

In this section we first describe the notation and then provide some elementary facts and notions on graphs, point mass dynamics, LMIs, and hybrid systems, which shall be used in the subsequent sections.

Figure 1: Formation Coordinates

2.2 Notation

Formation flying consists of flying a group of spacecraft in a particular pattern. To be able to express the time evolution of the formation and design the corresponding control laws, it is convenient that a reference frame is attached to each spacecraft. We shall always assume that these reference frames are induced from a dextral of three orthonormal vectors. Let the formation have n spacecraft labeled as $1, 2, \ldots, n$. Let $\mathcal{F}^i$ denote the reference frame attached to the i-th spacecraft; $\mathcal{F}^I$ on the other hand shall designate the inertial reference frame. For the inertia and the mass of the i-th spacecraft we use I^i and m^i, respectively. The force and torque acting upon i are denoted by f^i and τ^i; for the mass normalized force we used $u^i := \dfrac{f^i}{m^i}$. The time derivative with respect to $\mathcal{F}^i$ shall be denoted by $\frac{d}{dt_i}$; likewise, $\frac{d}{dt}$ will be used for the time derivative with respect to $\mathcal{F}^I$. r^{ij} denotes the position of the origin of $\mathcal{F}^i$ with respect to $\mathcal{F}^j$; r^i is the position of the origin of $\mathcal{F}^i$ with respect to $\mathcal{F}^I$. The desired position of the origin of $\mathcal{F}^i$ with respect to $\mathcal{F}^j$ shall be denoted by r_d^{ij}, and by r_d^i when $j = I$. The velocity of the origin of $\mathcal{F}^i$ with respect to $\mathcal{F}^j$, the velocity of the origin of $\mathcal{F}^i$ with respect to $\mathcal{F}^I$, the desired velocity of the origin of $\mathcal{F}^i$ with respect to $\mathcal{F}^j$, and the desired velocity of the origin of $\mathcal{F}^i$ with respect to $\mathcal{F}^I$, shall be denoted by v^{ij}, v^i, v_d^{ij}, and v_d^i, respectively. The vector $[r^i \; v^i]^T$ shall be referred to as the state of the i-th spacecraft and will be denoted by x^i. Similar notations are used for the attitude and the angular velocity of $\mathcal{F}^i$ with respect to $\mathcal{F}^j$: q^{ij} and ω^{ij} are the attitude and the angular velocity of $\mathcal{F}^i$ with respect to $\mathcal{F}^j$ and q_d^{ij} and ω_d^{ij} are the desired angular velocity and attitude of $\mathcal{F}^i$ with respect to $\mathcal{F}^j$ (refer to Figure 1). All other notations are standard: $\Re^n$ denotes the real Euclidean space of dimension n; $\|\cdot\|_\infty$ and $\|\cdot\|$ are used for the infinity norm and the 2-norm for

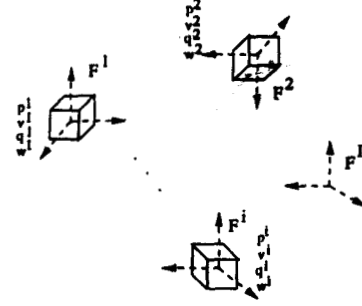vectors and matrices. The cross product matrix induced by the vector $x = [x_1 \, x_2 \, x_3]^T$ is the matrix,

$$[x] := \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

2.3 Elementary Dynamics: Inertial and Moving Frames

The dynamics of a point mass in an inertial reference frame is described by Newton second law as $\frac{d^2 r}{dt^2} = \frac{f}{m}$, where r, f, and m denote, respectively, the (inertial) position, the force, and the mass of the point mass. If the coordinate frame where positions and velocities are measured is itself rotating or accelerating, then the equations of motion are modified to reflect such a motion. Recall that the first and the second derivative of a vector A in $\mathcal{F}^I$ and $\mathcal{F}^i$ are related by the relation,

$$\frac{dA}{dt} = \frac{dA}{dt_i} + \omega^i \times A, \tag{2.1}$$

where ω^i is the angular velocity of $\mathcal{F}^i$ with respect to $\mathcal{F}^I$. In particular,

$$\frac{d\omega^i}{dt} = \frac{d\omega^i}{dt_i} + \omega^i \times \omega^i = \frac{d\omega^i}{dt_i}, \tag{2.2}$$

stating that the rate of change of the angular velocity is independent of the frame of reference where it is measured.

Differentiating both sides of (2.1) with respect to $\mathcal{F}^I$, we obtain,

$$\frac{d^2 A}{dt^2} = \frac{d^2 A}{dt_i^2} + \frac{d\omega^i}{dt_i} \times A + 2\omega^i \times \frac{dA}{dt_i}$$
$$+ \omega^i \times (\omega^i \times A). \tag{2.3}$$

In (2.3), the second, third, and fourth terms on the right hand side of the equality sign, are referred to as the angular, Coriolis, and centripetal acceleration, respectively.

One can therefore write the equation of motion in the moving reference frame $\mathcal{F}^i$ as,

$$\frac{d^2r}{dt_i{}^2} + \frac{d\omega^i}{dt_i} \times r + 2\omega^i \times \frac{dr}{dt_i} + \omega^i \times (\omega^i \times r)$$

$$= \frac{f^i}{m^i}. \quad (2.4)$$

2.4   LMIs and the State Feedback Design

Let $A \in \Re^{n \times n}$ and $B \in \Re^{n \times m}$ and consider the linear time invariant (LTI) system,

$$\dot{x} = Ax + Bu. \quad (2.5)$$

Suppose that a state feedback is to be synthesized which stabilizes (in addition to some other performance criteria) the LTI system (2.5) via the Lyapunov inequality: find K and P such that,

$$(A + BK)^T P + P(A + BK) < 0, \quad P > 0. \quad (2.6)$$

The system of inequalities (2.6) as it is written, is a very difficult computational problem (in fact it is in the form a bilinear matrix inequality (BMI) [10]). By a change of variable however, (2.6) can be written as an LMI. In this direction, let $Q = P^{-1}$ and perform a congruent transformation on (2.6) by Q; then (2.6) is transformed to,

$$QA^T + AQ + BY + Y^T B^T < 0, \quad Q > 0, \quad (2.7)$$

where we have set $Y = KQ$. The inequality (2.7) is now readily solvable by a host of efficient numerical methods, notably by the interior point methods [12]. In fact, the interior point methods can be used to even optimize an objective functional over the feasible set defined by (2.7),

$$\min_{Q,Y,\alpha} \alpha \quad (2.8)$$

$$QA^T + AQ + BY + Y^T B^T < \alpha I, \quad (2.9)$$

$$Q > 0; \quad (2.10)$$

an optimization problem in the form of (2.8)-(2.9) is referred to as a semi-definite program (SDP) [12].

When deriving control laws for each spacecraft in the formation we shall consider the situation where the control is always a function of the error in the state of the spacecraft; when this error is in fact the measured state by the controller, the LMI (2.7), or the SDP (2.8)-(2.10) can be conveniently used to come up with a stabilizing control law as outlined above.

2.5   Hybrid and Switching Systems

A hybrid dynamical system is a system whose time evolution is governed by a combination of logical (or discrete), as well as continuous variables [1], [5], [6], [8], [11], [13]. In particular, an autonomous hybrid system can be represented in the form of,

$$\dot{x} = f(x(t), \sigma(t)),$$
$$\sigma(t) = v(x(t), \sigma(t^-)),$$

where $x(t) \in \Re^n$, and $\sigma(t)$ belongs to a discrete set $\mathcal{N}$ (finite or infinite). Here for each σ, $f(., \sigma)$: $\Re^n \to \Re^n$ is a globally Lipschitz continuous function and $v : \Re^n \times \mathcal{N} \to \mathcal{N}$ describes the dynamics of the finite states. The notation $\sigma(t^-)$ indicates the value of the finite states a right before time t.

Similar to an autonomous hybrid system, a controlled hybrid system can be represented in the form,

$$\dot{x} = f(x(t), \sigma(t), u(t)), \quad (2.11)$$
$$\sigma(t) = v(x(t), \sigma(t^-), u(t)), \quad (2.12)$$

with the exception that $u(t) \in \Re^m$ and the definition of f and v are modified accordingly. It is not hard to see that switching between various controllers is in fact a special class of controlled hybrid systems; we present few such switching examples in the context of formation flying later on in the paper.

2.6   Elementary Graph Theory

A graph $G = (V, E)$ consists of a finite nonempty set V of vertices and a finite set of edges, E [3]. The cardinality of V is called the order of G. The graph G is called undirected if every edge is an unordered pair of distinct vertices; if the edges correspond to an ordered pair of distinct vertices then G is called directed. We shall represent an edge in a directed graph with the end vertices labeled as v and w, as $[v, w]$; similarly $\{v, w\}$ is used to denote the corresponding edge in an undirected graph. We say

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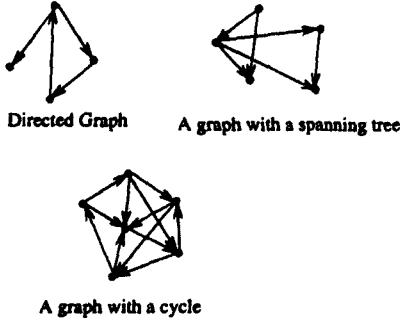Directed Graph   A graph with a spanning tree

A graph with a cycle

Figure 2: Some Graph Theoretic Concepts

that the edge $[v, w]$ (or $\{v, w\}$) is incident on v and w. A vertex which is not incident to any edge is called isolated. Two vertices that are connected by an edge are called adjacent vertices. The set of adjacent vertices of the vertex w is denoted by $\Gamma(w)$.

The degree of a vertex is the number of its adjacent vertices. A path from v^1 to v^k is the list of vertices $[v^1, v^2, \ldots, v^k]$ such that $[v^i, v^{i+1}] \in E$, for all $i = 1, \ldots, k-1$. If for every two vertices in a graph there is a path connecting them, then we call the graph connected. The path is called simple if all of its vertices are distinct. A path is a cycle if $k > 1$ and $v^1 = v^k$, and a simple cycle if in addition, $v^1, \ldots, v^{k-1}$ are distinct. A graph without a cycle is called acyclic.

We say that $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A tree is a connected graph that contains no cycles. A spanning tree of a graph G is a subgraph of G which is a tree and whose vertices are a subset of V; refer to Figure 2.

3 The Leader-Following (LF) Graph

Leader-following (LF) refers to a scenario where a reference frame, or a set of reference frames are designated such that the desired spacecraft positions in the formation are expressed relative to them.[1] Before we make the notion of LF more precise, we state the following definitions.

Definition 3.1 *The formation pattern of a group*

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[1] A group of spacecraft whose desired positions are expressed with respect to an inertial frame is thus an special case of LF.

of n spacecraft, $1, \ldots, n$, is the vector,

$$\mathcal{P}(t) := [r^1(t), \ldots, r^n(t)]^T \in \Re^{3n}.$$

Definition 3.2 *The formation configuration of a group of n spacecraft, $1, \ldots, n$, is the vector,*

$$\mathcal{C}(t) := [r^1(t), v^1(t), \ldots, r^n(t), v^n(t)]^T \in \Re^{6n}.$$

The desired formation pattern and configuration at time t can naturally be defined as,

$$\mathcal{P}_d(t) := [r_d^1(t), \ldots, r_d^n(t)]^T, \quad \text{and}$$
$$\mathcal{C}_d(t) := [r_d^1(t), v_d^1(t), \ldots, r_d^n(t), v_d^n(t)]^T;$$

such a specification requires that every spacecraft in the formation has knowledge of its inertial position and velocity at all times. However, what often is of interest in the context of formation flying is the *relative* spacecraft positions and velocities. We are thus led to express $\mathcal{P}_d(t)$ (or $\mathcal{C}_d(t)$) in such a way that for a set of indices $K \subseteq \{1, \ldots, n\}$, $r_d^k(t)$ is specified as a function of r^l, $h(r^l(t))$, for all $k \in K$ and some $l \in \{1, \ldots, n\} - K$, where,

$$h : \Re^3 \to \Re^3,$$

is an invertible map which is twice differentiable on a given time interval of interest. Given that h is an affine map, r_d^k can be expressed as,

$$r_d^k(t) = H^{kl} r^l(t) + h^{kl}(t), \qquad (3.13)$$

for a matrix $H^{kl} \in \Re^{3 \times 3}$ and $h^{kl}(t) \in \Re^3$. In the subsequent sections we shall mainly consider the situation where H is the *identity matrix*, although all of the results can be extended to the case where H is a nonsingular matrix.

Definition 3.3 i *is the leader of j if r_d^j is expressed as $h(r^i)$, for some invertible map $h : \Re^3 \to \Re^3$ which is twice differentiable on a given time interval of interest.*[2]

Note that LF is simply an assignment and reflects how one *decides* to represent the desired formation pattern and configuration. LF can conveniently be represented in terms of a graph as we now proceed to show.

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[2] Although the piece-wise diffeomorphicity of h is not necessary for the purpose of defining the leader-follower assignment, this qualification shall be used for deriving control laws in §5.

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Let $G_{LF} = (V_{LF}, E_{LF})$ be a directed graph of order n, with $V_{LF} = \{1, \ldots, n\}$ and $[i, j] \in E_{LF}$ if and only if i is the leader of j. We call G_{LF} the LF graph of the formation. Closely related to the LF graph is the communication graph, $G_C = (V_C, E_C)$. G_C is a directed graph with $V_C = V_{LF}$; however $[i, j] \in E_C$ if and only if i can send real valued messages to j via a communication protocol. At times, it would be more convenient to make G_C undirected, particularly when $[i, j] \in E_C$ automatically implies that $[j, i] \in E_C$.

The LF assignment is closely related to another concept which shall be referred to as dependency.

Definition 3.4 *j is dependent on i if u^j is a function of x^i.*

Dependency is the result of a situation where the control action of once spacecraft depends on the states (inertial positions and velocities) of another (set of) spacecraft in the formation.

Proposition 3.1 *If i is the leader of j, then j is dependent on i.*

Definition 3.5 *j is strongly dependent on i if u^j is a function of x^i and u^i.*

Associated with the set of dependent and strongly dependent vertices of G_{LF}, one can define the (directed) subgraphs $G_{LF/D}$ and $G_{LF/SD}$, respectively. The graphs G_{LF}, $G_{LF/D}$, and $G_{LF/SD}$, by the virtue of being constructed from the leader-follower assignments and the associated data dependencies, have various properties, few of which will be stated and proved below. Since in the subsequent sections we shall mainly study LF assignments which are defined by $G_{LF/SD}$, we focus on the properties of $G_{LF/SD}$ rather than G_{LF}.

Definition 3.6 *The LF assignment is called complete if the corresponding $G_{LF/SD}$ is connected.*

Proposition 3.2 *$G_{LF/SD}$ associated with a complete LF assignment contains a spanning tree.*

Definition 3.7 *A qualified subgraph of $G_{LF/SD}$ is any of its spanning trees.*

Proposition 3.3 *For a complete LF assignment associated with the formation of n spacecraft, there are at least $n - 1$ LF assignments.*

A complete LF assignment induces $G_{LF/SD}$ which contains more than one qualified subgraphs. However, one generally need only one of these subgraphs for the purpose of implementing the formation control law. We can thus consider a situation where there is a cost associated with every edge in $G_{LF/SD}$ (e.g., communication cost) and consider finding the most economical spanning tree. This topic shall be examined in more detail elsewhere, although it should be clear that can employ several existing algorithms for finding the most economical qualified subgraph of $G_{LF/SD}$.

4 Simple Control Laws for LF

In this section we go over some simple control laws which are derived based on the state feedback synthesis procedure discussed in §2.4 . These control laws can be used for the control of the formation pattern and configuration under two different measurement scenarios. First, we consider the situation where inertial measurements are available to both the leader(s) and the follower(s); then we comment on the case where the follower(s) measurements are done with respect to its own moving reference frame.

4.1 Inertial Reference Frame Measurements

Let i be the affine leader of j (with $H^{ji} = I$) during the time interval $[t_0, t_f]$. The desired position of j is thus expressed as,

$$r_d^j(t) = r^i(t) + h^{ij}(t), \quad t_0 \le t \le t_f.$$

The error expression for j is then simply,

$$e^j(t) = r_d^j(t) - r^j(t) = r^i(t) - r^j(t) + h^{ij}(t).$$

Assuming that h^{ij} is twice differentiable on $[t_0, t_f]$, the above expression can be differentiated twice with respect to the inertial reference frame to obtain (recalling that $u^i = \frac{f^i}{m^i}$),

$$\frac{d^2 e^j(t)}{dt^2} = u^i(t) - u^j(t) + \frac{d^2 h^{ij}(t)}{dt^2}. \qquad (4.14)$$

By letting,

$$u^j(t) = u^i + \frac{d^2 h^{ij}(t)}{dt^2} + \bar{v}^j(t), \qquad (4.15)$$

(4.24)-(4.25) can then be written as,

$$\dot{z}_1(t) = z_2(t), \quad (4.28)$$
$$\dot{z}_2(t) = W_1 z_1(t) + W_2 z_2(t)$$
$$+ u^i(t) - u^j(t), \quad (4.29)$$

where,

$$W_1 = z_3 z_3^T - \|z_3\|_2^2 I, \quad \text{and} \quad W_2 = -[2z_3].$$

Consider again the change of variable of the form,

$$u^j(t) = u^i(t) + \frac{d^2 h^{ij}(t)}{dt^2} + \bar{u}^j(t),$$

then,

$$\left[\begin{array}{c} \dot{z}_1(t) \\ \dot{z}_2(t) \end{array} \right] = \overbrace{\left[\begin{array}{cc} 0 & I \\ W_1 & W_2 \end{array} \right]}^{A^j} \left[\begin{array}{c} z_1(t) \\ z_2(t) \end{array} \right]$$
$$+ \underbrace{\left[\begin{array}{c} 0 \\ -I \end{array} \right]}_{B^j} \bar{u}^j(t). \quad (4.30)$$

Define the matrices A^j and B^j as suggested above. We can then proceed as in §4.1 and solve the LMI,

$$A^j Q + Q(A^j)^T + B^j Y + Y^T (B^j)^T < 0, (4.31)$$
$$Q > 0, (4.32)$$

and let,

$$u^j(t) = u^i(t) + \frac{d^2 h^{ij}(t)}{dt^2} + Y Q^{-1} z^j(t),$$
$$t_0 \leq t \leq t_f;$$

note that only the definition of the matrix A^j has been modified from §5.1 to reflect the fact that the error vector is now measured in the moving coordinate frame attached to the follower.

2. j has non-constant angular velocity: If the angular velocity of j does not remain constant during the LF, then we can use feedback linearization to linearize the dynamics in such a way that the LMI approach above can still be adopted. For this purpose it suffices to let,

$$a^j = -2z_3(t) \times z_2(t) - (I^j)^{-1}(\tau^j(t)$$
$$-z_3(t) \times I^j z_3(t)) - z_3(t)(z_3(t) \times z_1(t)), (4.33)$$

and let,

$$u^j(t) = u^i(t) + \frac{d^2 h^{ij}(t)}{dt^2} + \bar{u}^j(t) + a^j(t);$$

as before the expression for $\bar{u}^j(t)$ is found be solving the LMI (4.18)-(4.19).

In both scenarios considered above, the control law for the *leader* spacecraft i can also be based on the state feedback synthesis. For this purpose it suffices to let,

$$u^i(t) = Y Q^{-1} z(t) + \frac{d^2 r_d^i(t)}{dt^2},$$

where the matrices Y and Q are found from the LMI (4.18)-(4.19) by letting,

$$A^i = \left[\begin{array}{cc} 0 & I \\ 0 & 0 \end{array} \right], \quad \text{and} \quad B^i = \left[\begin{array}{c} 0 \\ -I \end{array} \right];$$

however z is now simply $r_d^i(t) - r^i(t)$.

5 Changing the LF Graph: Leadership Re-Assignment

The designation of the leader, aside from its associated hardware and software considerations and the required communication protocol, is rather arbitrary. It is thus of interest to consider a situation where the leader assignments are time varying. In this direction, we would like to study how the formation control performance is effected by changing the LF graph as,

$$G_{LF}^1 \rightarrow G_{LF}^2 \rightarrow G_{LF}^n \rightarrow \cdots$$

Consider for example a two spacecraft formation; let G_{LF}^k and G_{LF}^{k+1} be defined by,

$$G_{LF}^k = (\{i, j\}, [i, j]), \quad \text{and} \quad GL_{LF}^{k+1} = (\{i, j\}, [j, i]).$$

G_{LF}^k corresponds to the situation where i is the leader of j and therefore (assuming an affine leadership),

$$r_d^j(t) = r^i(t) + h^{ji}(t).$$

In this case the control law of §5.2 (when inertial measurements are available) can be implemented as,

$$u^i(t) = K z^i(t) + \frac{d^2 r_d^i(t)}{dt^2},$$
$$u^j(t) = K z^j(t) + u^i(t) + \frac{d^2 h^{ji}(t)}{dt^2},$$

where $z^i(t)$ is the state error observed by i at time t. Since $h^{ij}(t) = -h^{ji}(t)$, for G_{LF}^{k+1} the control law can be expressed as,

$$u^i(t) = K z^i(t) + u^j(t) + \frac{d^2 h^{ij}(t)}{dt^2} = K z^i(t)$$
$$+ u^j(t) - \frac{d^2 h^{ji}(t)}{dt^2},$$
$$u^j(t) = K z^j(t) + \frac{d^2 r_d^j(t)}{dt^2}.$$

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Figure 3: Switching for Leader Reassignment

6 LF Capturing

We consider a situation where a free spacecraft is captured by an LF; translated in terms of G_{LF}, this corresponds to the case where an isolated vertex is connected to G_{LF} (Figure 3). Note that we have considered the situation where the isolated spacecraft is not assigned as a leader; if this is in fact the case, then its control law will not be changed from when the spacecraft was free. However, in this latter situation, the control law for the new followers of the new leader spacecraft changes according to the procedure presented in §5.

7 Control Saturations

We now examine the scenario where the j-th spacecraft, j, following i in an LF, is also avoiding control saturation by switching between two or more controllers. In order to simplify the presentation, we shall assume in the rest of this section that,

$$\frac{d^2 h^{ji}(t)}{dt^2} = 0, \quad t_0 \le t \le t_f.$$

Recall that following the discussion of §4 (when inertial measurements are available), the state error dynamics of j can be represented by,

$$\dot{z}(t) = (A + BK)z(t),$$

where,

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -I \end{bmatrix}, \quad \text{and} \quad K = YQ^{-1};$$

the matrices Y and Q are found from solving the LMI,

$$AQ + QA^T + BY + Y^T B^T < 0, \quad Q > 0.$$

Let m denote the 2-norm of the maximum allowable mass normalized force on each spacecraft in the formation; that is, we require that,

$$\|u^j(t)\| \le m, \quad t_0 \le t \le t_f. \quad (7.34)$$

Note that although (7.34) imposes a constraint on the control vector expressed in the inertial frame, it

directly translates into a requirement expressed in the spacecraft body frame via an orthogonal transformation (recall the invariance of the 2-norm under an orthogonal transformation). Now since,

$$u^j(t) = YQ^{-1}z(t) + u^i(t),$$

we require that,

$$\|u^i(t) + YQ^{-1}z(t)\| \le m, \quad t_0 \le t \le t_f. \quad (7.35)$$

Note that although j has no prior knowledge about the values of u^i, it has to choose Q and Y such that (7.35) is satisfied. To cope with this lack of knowledge on the values of $u^i(t)$, we proceed to present a controller switching mechanism which satisfies the control constraint (7.35), in face of the lack of a priori knowledge of the values of $u^i(t)$ by the follower spacecraft. The only assumption which is required for the proposed approach to work is that,

$$\|u^i(t)\| < m, \quad t_0 \le t \le t_f.$$

Starting from (7.35), we work instead with the stronger requirement,

$$\|YQ^{-1}z(t)\| \le m - \|u^i(t)\| = m^i(t), \quad t_0 \le t \le t_f.$$

Let,

$$\mathcal{E}_{t_0} = \{z : z^T Q_{t_0}^{-1} z \le 1\},$$

where Q_{t_0} is a positive definite matrix which is chosen such that $z(0)$ belongs to $\mathcal{E}_{t_0}$, by solving the LMI,

$$\begin{bmatrix} 1 & z^T(0) \\ z(0) & Q_{t_0} \end{bmatrix} \ge 0, \quad Q_{t_0} > 0. \quad (7.36)$$

Suppose that we solve the LMI (7.36) in conjunction with,

$$AQ_{t_0} + Q_{t_0}A + BY_{t_0} + Y_{t_0}^T B^T < 0. \quad (7.37)$$

For small values of δt, if $x_d^j(t_0) = x_d^j(t_0 + \delta t)$ for $t \in [t_0, t_0 + \delta t]$ and we use the controller $K_{t_0} = Y_{t_0}Q_{t_0}^{-1}$, then it would be the case that $z(t_0 + \delta t) \in \mathcal{E}_{t_0}$. In fact, if $x_d^j(t)$ remains constant, then $z(t) \in \mathcal{E}_{t_0}$ for all $t \in [t_0, t_f]$. In this situation, in order to guarantee that the saturation constraint is not violated, we can augment the LMIs (7.36) and (7.37) with an LMI [4],

$$\begin{bmatrix} Q_{t_0} & Y_{t_0}^T \\ Y_{t_0} & m^i(t) \end{bmatrix} \ge 0, \quad (7.38)$$

Now, the problem is that in general, one cannot guarantee that $z(t_0 + \delta t) \in \mathcal{E}_{t_0}$, nor does the above

discussion addresses the situation where $m^i(t)$ does not remain constant. We are thus led to incorporate logic-based switching in conjunction with LMIs (7.36)-(7.38) to address both of these scenarios. Let,

$$\underline{m}^i := \min_{t \in [t_0, t_f]} \|m^i(t)\|;$$

solve the SDP,

$$\min_{Q_{t_0}, Y_{t_0}, \alpha} \alpha \quad (7.39)$$

$$AQ_{t_0} + Q_{t_0}A^T + BY_{t_0} + Y_{t_0}^T B^T < \alpha I, \quad (7.40)$$

$$Q_{t_0} > 0, \quad (7.41)$$

$$\begin{bmatrix} 1 & z^T(0) \\ z(0) & Q_{t_0} \end{bmatrix} \geq 0, \quad (7.42)$$

$$\begin{bmatrix} Q_{t_0} & Y_{t_0}^T \\ Y_{t_0} & \underline{m}^i \end{bmatrix} \geq 0, \quad (7.43)$$

$$\alpha < 0. \quad (7.44)$$

We shall assume, without loss of generality, that the above SDP is feasible. Let us now proceed from time t_0 and consider the various scenarios which can occur at time $t_0 + \delta t$:

1. $z(t_0 + \delta t) \in \mathcal{E}_{t_0}$ and $m^i(t)$ has remained constant: In this case it is guaranteed that,

$$\|K_{t_0} z(t_0 + \delta t)\| = \|Y_{t_0} Q_{t_0}^{-1} z(t_0 + \delta t)\|$$
$$\leq m^i(t + \delta t) = m^i(t);$$

thus the control constraint is not violated if the same control law used at time t_0 is applied at time $t_0 + \delta t$.

2. $z(t_0 + \delta t) \in \mathcal{E}_{t_0}$, however m^i has changed over the interval $[t_0, t_0 + \delta t]$: For this case we can proceed in two directions:

   (a) We consider the trajectory as staying in the ellipsoid $\mathcal{E}_{t_0}$; however we can modify the controller gain K_{t_0} by changing the matrix Y_{t_0}:

   $$\min_{Y_{t_0+\delta t}, \beta} \beta \quad (7.45)$$

   $$BY_{t_0+\delta t} + Y_{t_0+\delta t}^T B^T$$
   $$< -(AQ_{t_0} + Q_{t_0}A^T) + \beta I, \quad (7.46)$$

   $$\begin{bmatrix} Q_{t_0} & Y_{t_0+\delta t}^T \\ Y_{t_0+\delta t} & m^i(t_0 + \delta t) \end{bmatrix} \geq 0. \quad (7.47)$$

   The state feedback gain can now be set as,

   $$K_{t_0+\delta t} = Y_{t_0+\delta t} Q_{t_0}^{-1}. \quad (7.48)$$

Proposition 7.1 *The SDP (7.45)-(7.47) is feasible.*

(b) Given that $m^i(t_0 + \delta t) \geq \underline{m}^i$, we might be able to find a smaller ellipsoid where the error $z(t_0 + \delta t)$ belongs, by solving the following SDP:

$$\min_{Q_{t_0+\delta t}, Y_{t_0+\delta t}, \alpha} \alpha \quad (7.49)$$

$$AQ_{t_0+\delta t} + Q_{t_0+\delta t}A^T$$
$$+ BY_{t_0+\delta t} + Y_{t_0+\delta t}^T B^T < \alpha I, \quad (7.50)$$

$$Q_{t_0+\delta t} > 0, \quad (7.51)$$

$$\begin{bmatrix} 1 & z^T(t_0 + \delta t) \\ z(t_0 + \delta t) & Q_{t_0+\delta t} \end{bmatrix} \geq 0, \quad (7.52)$$

$$\begin{bmatrix} Q_{t_0+\delta t} & Y_{t_0+\delta t}^T \\ Y_{t_0+\delta t} & \underline{m}^i \end{bmatrix} \geq 0, \quad (7.53)$$

$$\alpha < 0, \quad (7.54)$$

$$Q_{t_0+\delta t} \geq Q_{t_0}. \quad (7.55)$$

Lemma 7.2 *Given that $z(t) \in \mathcal{E}_{t_0}$ for all $t \in [t_0, t_f]$, both of the above controller switching mechanism result in a globally asymptotically stable hybrid dynamical system which is guaranteed to satisfy the control constraint.*

(c) $z(t_0 + \delta t) \notin \mathcal{E}_{t_0}$, whether or not m^i has remained constant: This scenario arises when the error at time $t_0 + \delta t$ leaves the ellipsoid generated to bound it at time t_0. For this case, we proceed to solve a new SDP,

$$\min_{Q_{t_0+\delta t}, Y_{t_0+\delta t}, \alpha} \alpha \quad (7.56)$$

$$AQ_{t_0+\delta t} + Q_{t_0+\delta t}A^T + BY_{t_0}$$
$$+ \delta t + Y_{t_0+\delta t}^T B^T < \alpha I, \quad (7.57)$$

$$Q_{t_0+\delta t} > 0, \quad (7.58)$$

$$\begin{bmatrix} 1 & z^T(0) \\ z(0) & Q_{t_0+\delta t} \end{bmatrix} \geq 0, \quad (7.59)$$

$$\begin{bmatrix} Q_{t_0+\delta t} & Y_{t_0+\delta t}^T \\ Y_{t_0+\delta t} & \underline{m}^i \end{bmatrix} \geq 0, \quad (7.60)$$

$$\alpha < 0, \quad (7.61)$$

in conjunction with another LMI which shall make the *analysis* of the resulting switching mechanism more manageable,

$$0 < Q_{t_0+\delta t} \leq Q_{t_0}. \quad (7.62)$$

In this case we let,

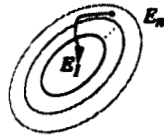$$K_{t_0+\delta t} = Y_{t_0+\delta t} Q_{t_0+\delta t}^{-1}.$$

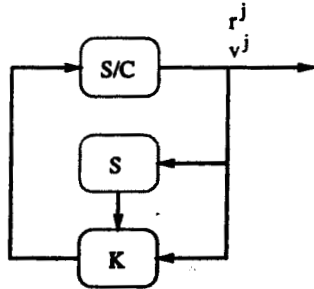Figure 4: Ellipsoids for Control Switching

Figure 5: S checks whether $z \in \mathcal{E}_{t_k}$ and switches the controller accordingly

Using this scheme, we get a sequence of positive definite matrices,

$$0 < Q_{t_n} \leq Q_{t_{n-1}} \leq \ldots \leq Q_{t_1} \leq Q_{t_0} \quad (7.63)$$

such that for all $t \in [t_0, t_f]$ and some t_k,

$$z(t) \in \mathcal{E}_{t_k} := \{z : z^T Q_{t_k}^{-1} \leq 1\}; \quad (7.64)$$

let k be the least index for which (7.64) holds. By the nested-ness property (7.63) such an index is unique.

Proposition 7.3 *The control switching mechanism proposed above results in a hybrid dynamical system where the origin is its globally asymptotically stable equilibrium point.*

8 Simulation Results

In this section we provide simulation results for few scenarios which demonstrate the types of switching described in the paper.

Figure 6 depicts the scenario where the leadership assignment is changed between the spacecraft in a two-spacecraft formation; the follower (either 1 or 2) both before and after the LF reassignment, is required to track a circular path with respect to leader.

Figure 7 demonstrates the LF capturing scenario, where a free spacecraft is captured by the LF and is
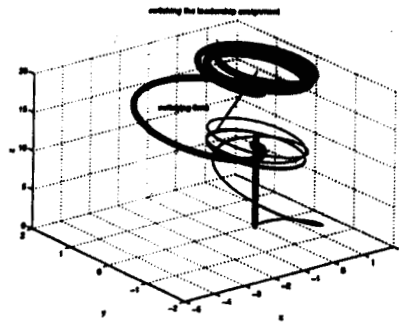

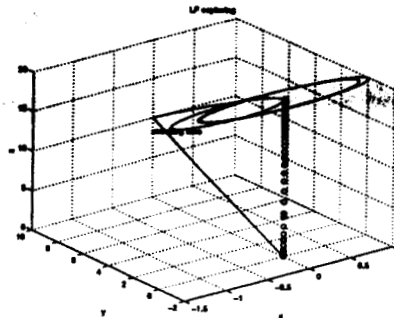Figure 6: Leadership reassignment

Figure 7: LF capturing

thus required to have a certain deviation from the leader after the capturing time. Finally, Figures 8-9 show an example where the follower's control law is switched in order to avoid control saturation as the result of a relatively high control input used by the leader.

9 Conclusion

We presented several new results on the formation flying control architecture based on the leader following strategy using ideas from graph theory,
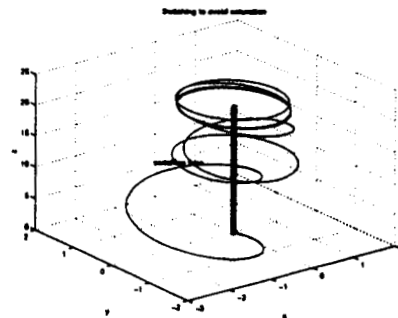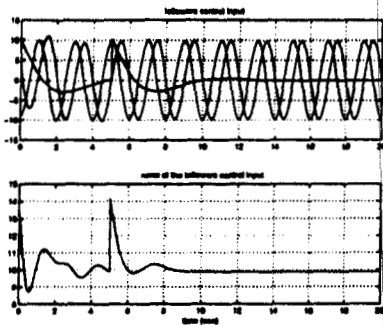

Figure 8: Control saturation prevention

Figure 9: follower switches controller to avoid saturation at $t = 5$ sec'

linear matrix inequalities, and logic-based switching. In this direction, the stability and the convergence properties of the resulting hybrid systems were given a particular attention.

Acknowledgment

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